Influence of a magnetic fluxon on the vacuum energy of quantum fields confined by a bag.

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Abstract

We study the simultaneous influence of boundary conditions and external fields on quantum fluctuations by considering vacuum zero-point energies for quantum fields in the presence of a magnetic fluxon confined by a bag, circular and spherical for bosons and circular for fermions. The Casimir effect is calculated in a generalized cut-off regularization after applying zeta-function techniques to eigenmode sums and using recent techniques about Bessel zeta functions at negative arguments.

1 Introduction

Aharonov-Bohm [1] settings may be regarded as one of the possible ways in which an external field modifies some observables of a given quantum system. In this specific phenomenon, the presence of an infinitely thin tube of magnetic flux alters the energy spectrum and brings about a modification of the vacuum energy, giving rise to a form of Casimir effect. Initially, Aharonov-Bohm fields acted on free particles[1]. The purpose of the present paper is to study the influence of the same type of fields on systems which are already constrained by boundary conditions (b.c.), working out the combined net effect of both on what would otherwise be a free system. The relevance of the Aharonov-Bohm scenario to some cosmic strings models including particles in the gravitational field of a spinning source is discussed in [2, 3]. A similar mathematical procedure is also applied to the description of Dirac fermions on black-hole backgrounds as shown in [4].

The quantum mechanical problem of a scalar particle inside a circular Aharonov-Bohm quantum billiard [6]-[9] (of radius a) bears a great resemblance, from the mathematical point of view, to the ones we set out to

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[1] For Casimir interactions between *two* solenoids see [5]

consider. In our case we have a classical magnetic fluxon which is coupled to a quantum field. We take this object to be an idealization of a vortex with a radially symmetric distribution of magnetic field in the limit where its characteristic thickness is vanishingly small. With this model in mind one has the physical basis to fix the boundary conditions at the origin to be imposed on the modes of the matter field, (a detailed account of this kind of analysis is given in [10]). We start with a complex, Klein-Gordon, massless field (the additional analytic effort which takes the treatment of a massive field by zeta function techniques is explained in [11]). We call ϕ the space-dependent part of the eigenmodes, which satisfies the equation (in units such that $\hbar = c = 1$)

$$\left(-i\vec{\nabla} - e\vec{A}\right)^2 \phi = \omega^2 \phi, \tag{1.1}$$

where the vector potential $\vec{A}$ is given by

$$e\vec{A}(\vec{r}) = \frac{\alpha}{r}\hat{e}_\varphi, \qquad \alpha = \frac{e\Phi}{2\pi}. \tag{1.2}$$

α is called *reduced flux*, being Φ the flux of the magnetic field. Since a billiard is a domain with perfectly reflecting walls, and we imagine an infinitely thin solenoid at the origin —reduced, in $D = 2$, to an unreachable point— the b.c. are $\phi = 0$ at $r = 0$ and $r = a$.

Zero-point energies emerge from mode-sums of the type $\frac{1}{2}\sum_n \omega_n$, and give rise to the Casimir effect [12]-[13]. Since the summation extends over all the ω_n's in the set of eigenmodes, such quantities usually diverge and need some regularization to make sense of them. To this end, we introduce spectral zeta functions as mere auxiliary tools, which will be denoted by

$$\zeta_{\mathcal{M}}(s) = \sum_n \omega_n^{-s}, \qquad \zeta_{\frac{\mathcal{M}}{\mu}}(s) = \sum_n \left(\frac{\omega_n}{\mu}\right)^{-s}. \tag{1.3}$$

μ is an arbitrary scale with mass dimensions, used to work with dimensionless objects. This is a regularization of analytical nature (see also [14]); other examples in this same category are the techniques in refs. [15] and refs. [16], [17]. When we discuss the results, we shall comment on their physical significance from the perspective of cut-off regularization. In this sense, our standpoint in the present case is that the zeta function is a purely mathematical object which affords a convenient method for the calculation of observables inasmuch as it may be connected with other, more physical, regularizations. Let us assume that we are using a general cut-off regularization for the vacuum energy which is given by

$$E_{reg} = \frac{1}{2}\sum_n \omega_n\, g\left(\frac{\omega_n}{\Lambda}\right), \tag{1.4}$$

where g is a well-behaved function which satisfies asymptotic expansions near the origin and at infinity of the following kind:

$$g(t) \sim 1 + \sum_{k=1}^{\infty} a_k t^k \ \ (t \to 0) \qquad g(t) \sim \frac{1}{t^{M_s}} \sum_{k=0}^{\infty} b_k t^{-k} \ \ (t \to \infty). \tag{1.5}$$

If we restrict this analysis for the sake of simplicity to a $(2+1)-D$ case, then g should be such that $M_s > 3$. This conditions guarantee that the Mellin transform of g has no poles in the strip $0 < \Re z < M_s$. The $\zeta_{\mathcal{M}}$-function, defined in (1.3), has its rightmost pole at $s = 2$. In fact, the requirement which we demand on the asymptotic behaviour of g might have been stated with greater generality. Essentially, g should go to 1 for small values of its

argument, and should go to zero for large values fast enough so that the following reasoning makes sense. Now we explain presently how the connection between the $\zeta_{\mathcal{M}}$-function (1.3) and expression (1.4) may be accomplished (for other discussions on how the results from different regulators may be related, see [21, 22]).

The Parseval formula for Mellin transforms allows us to write

$$E_{reg}(\Lambda) = \frac{1}{2}\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} dz\, \Lambda^{1+z}\, \mathrm{M}\left[g, z+1\right] \zeta_{\mathcal{M}}(z), \tag{1.6}$$

where r is any real number such that $2 < r < M_s - 1$. The structure of the poles of the integrand in expression (1.6) is closely related to the asymptotic expansion of $E_{reg}(\Lambda)$ for large values of the cut-off Λ. In the case at issue the relevant poles (the rightmost ones) are found at $z = 2$, $z = 1$ and $z = -1$. The first one is a simple pole due to the divergence of $\zeta_{\mathcal{M}}$ at $z = 2$. This induces the strongest divergence in E_{reg} (the one that is related to the volume (surface) and which in our two dimensional case goes as Λ^3). The next divergence arises from the pole of $\zeta_{\mathcal{M}}$ at $z = 1$ and goes as Λ^2. The last divergence stems from a pole at $z = -1$. This pole is in general a double one because both $\mathrm{M}\left[g, z+1\right]$ and $\zeta_{\mathcal{M}}$ have poles at that point. This fact means that by properly applying the Cauchy theorem, the integrand at $z = -1$ determines the finite part of E_{reg}, and also a divergent piece which goes as the logarithm of Λ. Let us give a more detailed account of these remarks. From the hypothesis we have stated, we may write

$$\begin{aligned}
\mathrm{M}\left[g,\, z\right] &= \frac{1}{z} + \mathrm{Fin}\left[\mathrm{M}\left[g,\, z=0\right]\right] + \mathcal{O}(z) \\
\zeta(z) &= \frac{\mathrm{Res}\left[\zeta,\, 2\right]}{z-2} + \mathcal{O}((z-2)^0) \\
\zeta(z) &= \frac{\mathrm{Res}\left[\zeta,\, 1\right]}{z-1} + \mathcal{O}((z-1)^0) \\
\zeta(z) &= \frac{\mathrm{Res}\left[\zeta,\, -1\right]}{z+1} + \mathcal{O}((z+1)^0),
\end{aligned} \tag{1.7}$$

where the symbol Fin means the extraction of the finite coefficient from the Laurent expansion of a function. Now we may use these expressions in (1.6) to get that, apart from terms which go to zero for large values of the cut-off Λ, the regularized expression for the vacuum energy is

$$\begin{aligned}
E_{reg}(\Lambda) &= \frac{1}{2}\left(\Lambda^3 \mathrm{M}\left[g,\, 3\right]\, \mathrm{Res}\left[\zeta,\, 2\right] + \Lambda^2 \mathrm{M}\left[g,\, 2\right]\, \mathrm{Res}\left[\zeta,\, 1\right]\right. \\
&\quad \left. + \ln\Lambda \mathrm{Res}\left[\zeta,\, -1\right] + \mathrm{Fin}\left[\mathrm{M}\left[g\right],\, z=0\right]\, \mathrm{Res}\left[\zeta,\, -1\right] + \mathrm{Fin}\left[\zeta,\, z=-1\right]\right).
\end{aligned} \tag{1.8}$$

The conclusion of this analysis is that to prove that the divergent terms in $E_{reg}(\Lambda)$ are independent of the magnetic flux, it suffices to show that the residues of $\zeta_{\mathcal{M}}$ at $z = 2$, $z = 1$ and $z = -1$ do not depend on this physical parameter. Let us put it another way, if we label the different heat-kernel coefficients B_0, $B_{\frac{1}{2}}$, B_1, $B_{\frac{3}{2}}$ ..., this independence boils down to saying that B_0, B_1 and $B_{\frac{3}{2}}$ do not depend on the magnetic flux. As for the finite part of expression (1.4) and the quantity $\mathrm{Fin}\left[\zeta,\, z=-1\right]$, their relationship is quite direct, as it appears explicitly in expression (1.8).

We shall systematically throw away such divergences which do not depend on α. These divergences would be relevant in a study about the bag dynamics (see [20, 11]), that is, concerned with situations where the bag walls are liable to deformation. In our case, we take the bag to be a perfectly rigid object.

In the free case (i.e. without flux) the eigenfrequencies under our conditions are zeros of J_ν Bessel functions with integer indices ν coming from angular momentum. The solutions for nonzero α have been found in [6, 9],

and basically correspond to an index shift with respect to the free case $|l| \to |l-\alpha|$. Since in both cases the eigenmodes are zeros of the same type of functions, we shall introduce the following 'partial-wave' zeta functions for fixed values of ν:

$$\zeta_\nu(s) = \sum_{n=1}^{\infty} j_{\nu,n}^{-s}\,, \text{ for Re } s > 1, \tag{1.9}$$

where $j_{\nu,n}$ denotes the nth nonvanishing zero of J_ν (see also [26, 27]; discrete versions of the Bessel problem, their solutions and associated zeta functions have also been studied in [32]).

When considering the whole problem in a D-dimensional space, one must take into account the degeneracy $d(D,l)$ of each angular mode in D dimensions. Therefore, we define the 'complete' spherical zeta function

$$\zeta_{\mathcal{M}}(s) \;=\; a^s \sum_{l=l_{\min}}^{\infty} d(D,l) \sum_{n=1}^{\infty} j_{\nu(D,l),n}^{-s} \;=\; a^s \sum_{l=l_{\min}}^{\infty} d(D,l)\, \zeta_{\nu(D,l)}(s), \tag{1.10}$$

$l_{\min}$ is the minimum value (if any) of l. In the free case $\nu(D,l) = l + D/2 - 1$ and the general form of $d(D,l)$ (see e.g. [33]) is $d(D,l) = (2l+D-2)\dfrac{(l+D-3)!}{l!(D-2)!}$, but this will change when a flux is present.

Following the programme we have just put forward, the partial wave zeta function for scalars is obtained in sect. 2. From this starting point, we construct the complete zeta functions for $D=2$ and $D=3$ complex scalar fields in sects. 3 and 4, respectively, finding their analytic continuations to $s=-1$. Numerical results for the zero-point energy are then discussed. Afterwards, in sect. 5 we study the Dirac field in $D=2$, and sect. 6 is devoted to the conclusions.

2 'Partial-wave' zeta function

Computing the Casimir energy through the calculation of the complete zeta function requires the knowledge of the Bessel zeta functions (1.9) at $s=-1$, while the complex domain where (1.9) holds is bounded by Re $s=1$. This is a serious difficulty, but we know that $\zeta_\nu(s)$ admits an analytic continuation to other values of s. Moreover, in refs. [26] and [27] we showed how to obtain an integral representation of this continuation valid for $-1 <$ Re $s < 0$, which reads

$$\zeta_\nu(s) = \frac{s}{\pi} \sin\frac{\pi s}{2} \int_0^\infty dx\, x^{-s-1} \ln\left[\sqrt{2\pi x}\, e^{-x} I_\nu(x)\right], \quad \text{for } -1 < \text{Re } s < 0. \tag{2.1}$$

Whenever $\nu \neq 0$ we can work out (2.1) by the method explained in [29, 30] (see also [31] and [24]), arriving at

$$\begin{aligned}\zeta_\nu(s) \;=\;& \frac{1}{4}\sigma_1 \nu^{-s} \\ &+\nu^{-s}\frac{s}{\pi}\sin\frac{\pi s}{2}\left[\sigma_2\left\{\frac{1}{2s}B\left(\frac{s+1}{2},-\frac{s}{2}\right)+2^{s-1}B\left(\frac{s+1}{2},-s\right)\right.\right. \\ &\qquad\qquad \left.+2^{s-1}B\left(\frac{s+3}{2},-s\right)\right\}\nu \\ &\qquad \left.+\mathcal{S}_N(s,\nu)+\frac{1}{2}\rho B\left(\frac{s+1}{2},-\frac{s}{2}\right)\frac{1}{\nu}+\overline{\mathcal{J}}_1(s)\frac{1}{\nu}+\sum_{n=2}^{N}\mathcal{J}_n(s)\frac{1}{\nu^n}\right].\end{aligned} \tag{2.2}$$

with

$$\sigma_1 = -1, \qquad \sigma_2 = 1, \qquad \rho = \frac{1}{8}. \tag{2.3}$$

In addition

$$
\begin{aligned}
\mathcal{S}_N(s,\nu) &\equiv \int_0^\infty dx\, x^{-s-1}\left\{\ln\left[\mathcal{L}(\nu,x)\right]-\sum_{n=1}^{N}\frac{\mathcal{U}_n(t(x))}{\nu^n}\right\},\\
\mathcal{L}(\nu,x) &= \sqrt{2\pi\nu}(1+x^2)^{1/4}e^{-\nu\eta(x)}I_\nu(\nu x), \qquad \eta(x)=\sqrt{1+x^2}+\ln\frac{x}{1+\sqrt{1+x^2}},
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
\mathcal{U}_1(t) &= \frac{t}{8}-\frac{5\,t^3}{24},\\
\mathcal{U}_2(t) &= \frac{t^2}{16}-\frac{3\,t^4}{8}+\frac{5\,t^6}{16},\\
\mathcal{U}_3(t) &= \frac{25\,t^3}{384}-\frac{531\,t^5}{640}+\frac{221\,t^7}{128}-\frac{1105\,t^9}{1152},\\
\mathcal{U}_4(t) &= \frac{13\,t^4}{128}-\frac{71\,t^6}{32}+\frac{531\,t^8}{64}-\frac{339\,t^{10}}{32}+\frac{565\,t^{12}}{128},\\
&\vdots
\end{aligned}
\tag{2.5}
$$

the key point being that, this way, $S_N(s,\nu)$ is a *finite* integral at $s=-1$. Further,

$$
\begin{aligned}
\overline{\mathcal{J}}_1(s) &= -\frac{5}{48}B\left(\frac{s+3}{2},-\frac{s}{2}\right)\\
\mathcal{J}_n(s) &= \int_0^\infty dx\, x^{-s-1}\,\mathcal{U}_n(t(x)), \qquad t(x)=\frac{1}{\sqrt{1+x^2}},
\end{aligned}
\tag{2.6}
$$

Thus, the expressions for the $\mathcal{J}_n(s)$'s are easily obtained from the $\mathcal{U}_n(t)$'s in (2.5). In fact, since

$$
\int_0^\infty dx\, x^{-s-1}\left[t(x)\right]^m=\frac{1}{2}B\left(\frac{s+m}{2},-\frac{s}{2}\right),
\tag{2.7}
$$

the result of the x-integration is like replacing

$$
\begin{aligned}
\mathcal{U}_n(t) &\to \mathcal{J}_n(s)\\
t^m &\to \frac{1}{2}B\left(\frac{s+m}{2},-\frac{s}{2}\right).
\end{aligned}
\tag{2.8}
$$

Expression (2.2) is not valid for $\nu=0$, since it was obtained from a rescaling $x\to\nu x$ and subsequent application of uniform asymptotic expansions in νx. Moreover, numerically speaking it is little convenient if ν is very small. An alternative representation valid in these conditions is needed. Starting from (2.1), we subtract and add the asymptotic behaviour of the integrand, which gives rise to a logarithmic divergence on integration. When doing so, we shall write the large-x expansion of $\ln[\ldots]$ as follows:

$$
\ln\left[\sqrt{2\pi x}\,e^{-x}I_\nu(x)\right]=-\frac{4\nu^2-1}{8x}+\mathcal{O}\left(\frac{1}{x^2}\right)=-\frac{4\nu^2-1}{8\sqrt{x^2+1}}+\mathcal{O}\left(\frac{1}{x^2+1}\right).
\tag{2.9}
$$

Thus, the piece we separate can be integrated with the help of (2.7) ($m=1$ case) and we are left with

$$
\begin{aligned}
\zeta_\nu(s) &= \frac{s}{\pi}\sin\frac{\pi s}{2}\left[\mathcal{R}_\nu(s)-\frac{4\nu^2-1}{16}B\left(\frac{s+1}{2},-\frac{s}{2}\right)\right],\\
\mathcal{R}_\nu(s) &= \int_0^\infty dx\, x^{-s-1}\left\{\ln\left[\sqrt{2\pi x}\,e^{-x}I_\nu(x)\right]+\frac{4\nu^2-1}{8\sqrt{x^2+1}}\right\}.
\end{aligned}
\tag{2.10}
$$

Since the above integral is now finite at $s=-1$ we can Laurent-expand without problems around $s=-1$, arriving at

$$
\zeta_\nu(s)=\frac{1-4\nu^2}{8\pi}\frac{1}{s+1}+\frac{1-4\nu^2}{8\pi}(-1+\ln 2)+\frac{1}{\pi}\mathcal{R}_\nu(-1)+\mathcal{O}(s+1).
\tag{2.11}
$$

In particular, for $\nu=0$, $\mathcal{R}_0(-1)=-0.00723$ and

$$
\zeta_0(s)=\frac{1}{8\pi}\frac{1}{s+1}-0.01451+\mathcal{O}(s+1).
\tag{2.12}
$$

Note also the vanishing of the $s=-1$ pole when $\nu=\pm 1/2$ already explained in ref. [27].

3 $D = 2$ bosons

3.1 'Complete' zeta function

Next, we go on to the two-dimensional problem. Following ref.[9], one realizes that the eigenmode sum for this case gives rise to the following complete spectral zeta function

$$\zeta_{\mathcal{M}}(s;\alpha) = a^s \sum_{l=-\infty}^{\infty} \zeta_{|l-\alpha|}(s), \tag{3.1}$$

Since this function has the properties

$$\begin{aligned} \zeta_{\mathcal{M}}(s;\alpha+k) &= \zeta_{\mathcal{M}}(s;\alpha), \;\; k \in \mathbf{Z}, \\ \zeta_{\mathcal{M}}(s;-\alpha) &= \zeta_{\mathcal{M}}(s;\alpha), \end{aligned} \tag{3.2}$$

it is enough to study it for $0 \le \alpha \le 1/2$. Introducing

$$\overline{\zeta_{\mathcal{M}}}(s;\beta) \equiv a^s \sum_{l=0}^{\infty} \zeta_{l+\beta}(s), \tag{3.3}$$

we can write

$$\begin{aligned} \zeta_{\mathcal{M}}(s;\alpha) &= \overline{\zeta_{\mathcal{M}}}(s;\alpha) + \overline{\zeta_{\mathcal{M}}}(s;1-\alpha) \\ &= a^s \zeta_{|\alpha|}(s) + \overline{\zeta_{\mathcal{M}}}(s;1+\alpha) + \overline{\zeta_{\mathcal{M}}}(s;1-\alpha). \end{aligned} \tag{3.4}$$

Next we insert expression (2.2) into (3.3) and, realizing that $\sum_{l=0}^{\infty}(l+\beta)^{-s} = \zeta_H(s,\beta)$, where ζ_H stands for the Hurwitz zeta function, we find

$$\begin{aligned} \overline{\zeta_{\mathcal{M}}}(s;\beta) &= \frac{1}{4}\sigma_1 a^s \zeta_H(s,\beta) \\ &\quad + a^s \frac{s}{\pi} \sin\frac{\pi s}{2} \left[\sigma_2 \left\{ \frac{1}{2s} B\left(\frac{s+1}{2}, -\frac{s}{2}\right) + 2^{s-1} B\left(\frac{s+1}{2}, -s\right) \right.\right. \\ &\qquad\qquad \left. + 2^{s-1} B\left(\frac{s+3}{2}, -s\right) \right\} \zeta_H(s-1,\beta) \\ &\qquad + \sum_{l=0}^{\infty} \mathcal{S}_N(s, l+\beta)(l+\beta)^{-s} \\ &\qquad + \frac{1}{2}\rho B\left(\frac{s+1}{2}, -\frac{s}{2}\right) \zeta_H(s+1,\beta) \\ &\qquad \left. + \overline{\mathcal{J}}_1(s)\zeta_H(s+1,\beta) + \sum_{n=2}^{N} \mathcal{J}_n(s)\zeta_H(s+n,\beta) \right], \end{aligned} \tag{3.5}$$

with the values of σ_1, σ_2 and ρ in (2.3). Taking $N = 4$ and Laurent-expanding near $s = -1$, this may be written

$$\begin{aligned} \overline{\zeta_{\mathcal{M}}}(s;\beta) = \frac{1}{a} &\left[-\frac{1}{4}\zeta_H(-1,\beta) \right. \\ &+ \frac{1}{\pi} \left\{ \frac{1}{4}\zeta_H(-2,\beta) - \frac{5}{24}\zeta_H(0,\beta) - \frac{229}{40320}\zeta_H(2,\beta) + \frac{35}{65536}\zeta_H(3,\beta) \right. \\ &\quad + \sum_{l=0}^{\infty} S_4(-1, l+\beta)(l+\beta) \\ &\quad + \left(-\frac{\pi}{256} - \frac{1}{2}\zeta_H(-2,\beta) + \frac{1}{8}\zeta_H(0,\beta) \right) \left(\frac{1}{s+1} + \ln a - 1 \right) \\ &\quad - \frac{\pi}{64} + \frac{\ln 2}{16} - \beta\frac{\ln 2}{8} + \frac{\pi\psi(\beta)}{256} - \left(1 + \frac{1}{2}\ln 2\right)\zeta_H(-2,\beta) \\ &\quad \left.\left. - \frac{1}{2}\zeta'_H(-2,\beta) + \frac{1}{8}\zeta'_H(0,\beta) \right\} + \mathcal{O}(s+1) \right]. \end{aligned} \tag{3.6}$$

Concerning the pole at $s = -1$ of the complete zeta function, by (3.4), (2.11) and (3.6), and noticing that $\zeta_H(-2, 1+\alpha) + \zeta_H(-2, 1-\alpha) = -\alpha^2$, one arrives at

$$\zeta_{\mathcal{M}}(s;\alpha) = \frac{1}{a}\left[-\frac{1}{128}\frac{1}{s+1} + \mathcal{O}((s+1)^0)\right], \tag{3.7}$$

i.e. the residue is independent of α. The reader may check by using the method explained in ref.[24] that this independence applies not only to $B_{\frac{3}{2}}$, but also to B_0 and $B_{\frac{1}{2}}$. As we have explained in the introduction, this property allows us to state that in cut-off regularization the dependence of the vacuum energy on the magnetic flux does not appear in the divergent terms. The dependence of the vacuum energy on the magnetic flux is completely contained in the finite part of $\zeta_{\mathcal{M}}$.

Since we plan to use the same three formulas for calculating the finite parts, it will be necessary to obtain $\zeta'_H(0,\beta)$ and $\zeta'_H(-2,\beta)$ around $\beta = 1$. The first is known (see e.g. [34]) and amounts to

$$\zeta'_H(0,\beta) = \ln\Gamma(\beta) - \frac{1}{2}\ln(2\pi), \tag{3.8}$$

while the second is calculated by numerical evaluation of $\zeta'_H(-n,\beta)$ from an integral representation of the derivative of ζ_H valid for negative first arguments.

3.2 Numerical results

We start by the $l = 0$ partial wave zeta-functions obtained from (2.11). Since we are supposing $\alpha \geq 0$, the results will be denoted by

$$a^s\zeta_\alpha(s) = \frac{1}{a}\left[r_\alpha\left(\frac{1}{s+1} + \ln a\right) + p_\alpha + \mathcal{O}(s+1)\right], \qquad r_\alpha = \frac{1-4\alpha^2}{8\pi}, \tag{3.9}$$

where the finite parts p_α are listed in the second column of table 1. The pole absence for $\alpha = 1/2$ may be

α	p_α	$\bar{p}_{1+\alpha}$	$\bar{p}_{1-\alpha}$	q_α
0	-0.01451	$+0.01174$	$+0.01174$	$+0.00899$
0.1	-0.05971	$+0.04062$	-0.01172	-0.03081
0.2	-0.10771	$+0.07491$	-0.02987	-0.06266
0.3	-0.15778	$+0.11462$	-0.04285	-0.08601
0.4	-0.20932	$+0.15968$	-0.05095	-0.10060
$\frac{1}{2}$	$-\frac{\pi}{12} = -0.26180$	$+0.21001$	-0.05471	-0.10650

Table 1: Finite parts at $s = -1$ of the involved zeta funcions (for $a = 1$). Column 2: $l = 0$ partial wave zeta function $\zeta_\alpha(s)$ Columns 3 and 4: $\overline{\overline{\zeta_{\mathcal{M}}}}(s;\beta)$ with $\beta = 1+\alpha$ and $\beta = 1-\alpha$. Column 5: finite part of the complete zeta function $\zeta_{\mathcal{M}}(s;\alpha)$.

regarded as a consequence of the fact that $J_{1/2}(x) \propto \sin x$, and therefore $\zeta_{1/2}(x) = \pi^{-s}\zeta_R(s)$ (ζ_R meaning the Riemann zeta function), which is finite at $s = -1$ because $\zeta_R(-1) = -1/12$. Next, we find $\overline{\zeta_{\mathcal{M}}}(s;\beta)$ from (3.6) for the corresponding $\beta = 1 \pm \alpha$'s. We shall employ the notation

$$\overline{\zeta_{\mathcal{M}}}(s;\beta) = \frac{1}{a}\left[\bar{r}_\beta\left(\frac{1}{s+1} + \ln a\right) + \bar{p}_\beta\right] + \mathcal{O}(s+1). \tag{3.10}$$

According to (3.6)

$$\bar{r}_\beta = \frac{1}{\pi}\left[-\frac{\pi}{256} - \frac{1}{2}\zeta_H(-2,\beta) + \frac{1}{8}\zeta_H(0,\beta)\right] \tag{3.11}$$

(note that in terms of Bernoulli polynomials $\zeta_H(-n,x) = -\frac{1}{n+1}B_{n+1}(x)$). As for $\bar{p}_\beta$, we list some of its values in columns 3 and 4 of table 1. Using now (3.4) and the above results we get

$$\zeta_{\mathcal{M}}(s;\alpha) = \frac{1}{a}\left[-\frac{1}{128}\left(\frac{1}{s+1} + \ln a\right) + q_\alpha\right] + \mathcal{O}(s+1), \tag{3.12}$$

The already remarked α-indepedence of the resdiue is exhibited by the fact that $r_\alpha + \bar{r}_{1+\alpha} + \bar{r}_{1-\alpha} = -\dfrac{1}{128}$. Values of $q_\alpha = p_\alpha + \bar{p}_{1+\alpha} + \bar{p}_{1-\alpha}$ for different α's between 0 and 1/2 are given in the fifth column of table 1 (see also Fig. 1). Now it would be incorrect to say that the dependence of $E_{reg}(\Lambda)$ on α is exactly given by $\frac{1}{2a}\,q_\alpha$. We should take into account a factor 2 which stems from the complex nature of the scalar field. In other words, the dependence of $E_{reg}(\Lambda)$ on α is given by $\frac{1}{a}\,q_\alpha$, apart from terms which vanish when Λ goes to infinity.

4 $D = 3$ bosons

Eq. (1.1) is again considered, but now in $D = 3$ and with a magnetic flux line diametrically threading a sphere of radius a. We make such a gauge choice that the vector potential in spherical coordinates reads

$$e\vec{A}(\vec{r}) = \frac{\alpha}{r\sin\theta}\hat{e}_\varphi. \tag{4.1}$$

The spectrum and eigenfuctions for the associated quantum-mechanical problem have been written down by the authors in ref.[26]. The proof that one must impose regularity at the origin may be carried out as described in ref.[10]. After studying their associated degeneracies, we are able to write the complete zeta function as follows

$$\zeta_{\mathcal{M}}(s;\alpha) = a^s \sum_{p=0}^{\infty}\sum_{m=-\infty}^{\infty} \zeta_{|m-\alpha|+p+1/2}(s), \tag{4.2}$$

again, it is apparent that

$$\zeta_{\mathcal{M}}(s;\alpha+k) = \zeta_{\mathcal{M}}(s;\alpha) \tag{4.3}$$

for any integer k, and

$$\zeta_{\mathcal{M}}(s;1-\alpha) = \zeta_{\mathcal{M}}(s;\alpha). \tag{4.4}$$

It is now an immediate result that

$$\zeta_{\mathcal{M}}(s;\alpha) = \zeta_{\mathcal{M}}(s;-\alpha) \tag{4.5}$$

From (4.3) and (4.4) we may also restrict our study to the domain $0 \le \alpha \le \frac{1}{2}$, this is a property which we proceed to take advantge of in the sequel. Under this restriction we may give an alternative representaion for (4.2):

$$\zeta_{\mathcal{M}}(s;\alpha) = a^s \sum_{l=-\infty}^{\infty} |l|\,\zeta_{|l-\alpha+1/2|}(s). \tag{4.6}$$

In terms of

$$\widetilde{\zeta_{\mathcal{M}}}(s;\beta) \equiv a^s \sum_{l=0}^{\infty} l\,\zeta_{l+\beta}(s) \tag{4.7}$$

and of the $\overline{\zeta_{\mathcal{M}}}$ function defined in (3.3), $\zeta_{\mathcal{M}}(s;\alpha)$ reads

$$\zeta_{\mathcal{M}}(s;\alpha) = \overline{\zeta_{\mathcal{M}}}\left(s;\frac{1}{2}+\alpha\right) + \widetilde{\zeta_{\mathcal{M}}}\left(s;\frac{1}{2}+\alpha\right) + \widetilde{\zeta_{\mathcal{M}}}\left(s;\frac{1}{2}-\alpha\right). \tag{4.8}$$

Next, let's consider the relation between them and the new zeta function

$$\overline{\overline{\zeta_{\mathcal{M}}}}(s;\beta) \equiv a^s \sum_{l=0}^{\infty}(l+\beta)\zeta_{l+\beta}(s). \tag{4.9}$$

This has the advantage that $\overline{\overline{\zeta_{\mathcal{M}}}}(s;\beta)$ can be immediately found from known material. The case without magnetic flux has been already studied for $D=3$ in [29, 30]. Since $d(3,l)=2l+1=2\nu(3,l)$, formula (1.10) is now rewritten as $\zeta_{\mathcal{M}}(s;\alpha=0)=2a^s\sum_{l=0}^{\infty}\nu(3,l)\zeta_{\nu(3,l)}(s)$. Therefore, the expression for $\overline{\overline{\zeta_{\mathcal{M}}}}(s;\beta)$ is the one for the $\zeta_{\mathcal{M}}(s)$ in those works but for the simple replacement

$$\overline{\overline{\zeta_{\mathcal{M}}}}(s;\beta) = \frac{1}{2}\,\zeta_{\mathcal{M}}(s;\alpha=0)\left\{\nu(l)=(l+1/2)\longrightarrow(l+\beta)\right\}. \tag{4.10}$$

Thus, for $N=4$ subtractions we find

$$\begin{aligned}\overline{\overline{\zeta_{\mathcal{M}}}}(s;\beta) = \frac{1}{a}\quad &\left[-\frac{1}{256}\zeta_H(0,\beta)+\frac{1}{4\pi}\zeta_H(-3,\beta)-\frac{1}{4}\zeta_H(-2,\beta)-\frac{5}{24\pi}\zeta_H(-1,\beta)+\frac{35}{65536}\zeta_H(2,\beta)\right.\\ &+\frac{1}{\pi}\left\{\left(-\frac{229}{40320}-\frac{1}{2}\zeta_H(-3,\beta)+\frac{1}{8}\zeta_H(-1,\beta)\right)\left(\frac{1}{s+1}+\ln a-1\right)\right.\\ &\quad+\sum_{l=0}^{\infty}\mathcal{S}_4(-1,l+\beta)(l+\beta)^2\\ &\quad+\frac{293}{24192}-\frac{229}{40320}(\ln 2-\psi(\beta))\\ &\quad+\left(-1-\frac{1}{2}\ln 2\right)\zeta_H(-3,\beta)-\frac{1}{2}\zeta_H'(-3,\beta)\\ &\quad\left.\left.+\frac{1}{8}\ln 2\,\zeta_H(-1,\beta)+\frac{1}{8}\zeta_H'(-1,\beta)\right\}+\mathcal{O}(s+1)\right].\end{aligned} \tag{4.11}$$

As a result of the previous definitions,

$$\widetilde{\zeta_{\mathcal{M}}}(s;\beta) = \overline{\overline{\zeta_{\mathcal{M}}}}(s;\beta) - \beta\overline{\zeta_{\mathcal{M}}}(s;\beta). \tag{4.12}$$

Hence, the complete zeta function (4.8) is conveniently put in the way

$$\zeta_{\mathcal{M}}(s;\alpha) = \left(\frac{1}{2}-\alpha\right)\left[\overline{\zeta_{\mathcal{M}}}\left(s;\frac{1}{2}+\alpha\right)-\overline{\zeta_{\mathcal{M}}}\left(s;\frac{1}{2}-\alpha\right)\right]+\overline{\overline{\zeta_{\mathcal{M}}}}\left(s;\frac{1}{2}+\alpha\right)+\overline{\overline{\zeta_{\mathcal{M}}}}\left(s;\frac{1}{2}-\alpha\right), \tag{4.13}$$

and the necessary knowledge about the objects on the r.h.s. is available. We have already found the residue of $\overline{\zeta_{\mathcal{M}}}(s;\beta)$ at $s=-1$, namely

$$\mathrm{Res}\left[\overline{\zeta_{\mathcal{M}}}(s;\beta);s=-1\right] = \frac{1}{a}\bar{r}_\beta, \tag{4.14}$$

where $\bar{r}_\beta$ is the one in (3.11). Similarly, from (4.11)

$$\mathrm{Res}\left[\overline{\overline{\zeta_{\mathcal{M}}}}(s;\beta);s=-1\right] = \frac{1}{a\pi}\left[-\frac{229}{40320}-\frac{1}{2}\zeta_H(-3,\beta)+\frac{1}{8}\zeta_H(-1,\beta)\right]. \tag{4.15}$$

(At this point, one can check that

$$\mathrm{Res}\left[\overline{\overline{\zeta_{\mathcal{M}}}}(s;\beta=1/2);s=-1\right] = \frac{1}{a}\frac{1}{315\pi} = \mathrm{Res}\left[\frac{1}{2}\zeta_{\mathcal{M}}(s;\alpha=0);s=-1\right],$$

as should be). Then, (4.13) yields

$$\mathrm{Res}\,[\zeta_{\mathcal{M}}(s;\alpha); s=-1] = \frac{1}{a\pi}\left[\frac{2}{315} - \frac{1}{6}\alpha(1-\alpha^2)\left(1-\frac{\alpha}{2}\right)\right]. \tag{4.16}$$

We recall that this is valid for $0 \le \alpha \le \frac{1}{2}$, and one has to make use of (4.3) and (4.4) to extend it to any value, in particular, when we extend expression (4.16) to any real α, we have a non-analytic function of α. The theory cannot be renormalized. The situation is different if we give up the idea of a purely confining enclosure and allow the presence of external modes, but satisfying the same b.c. as the internal ones. Parallelling the steps in ref.[30] for the $\alpha = 0$ case, we construct the α-dependent complete zeta function for these external Dirichlet modes —say $\zeta_{\mathcal{M}\,\mathrm{ext}}(s;\alpha)$. With respect to the internal case, we have the following modifications:

$$\mathcal{L}(\nu, x) \to \sqrt{\frac{2\nu}{\pi}}(1+x^2)^{1/4} e^{\nu\eta(x)} K_\nu(\nu x), \tag{4.17}$$

and the ν-series undergoes a ν-parity change, which brings about the transformations

$$\begin{array}{lcl} \sigma_2 & \to & -\sigma_2, \\ \rho & \to & -\rho, \\ \mathcal{U}_n(t) & \to & (-1)^n \mathcal{U}_n(t), \\ \overline{\mathcal{J}}_1(s), \mathcal{J}_n(s) & \to & -\overline{\mathcal{J}}_1(s), (-1)^n \mathcal{J}_n(s). \end{array} \tag{4.18}$$

The external ζ-function ensuing from these transformations takes into account an overall subtraction from the Minkowsky space (for instance, the residue of the rightmost pole is negative). From the construction of this external ζ-function, all the terms contributing to the $s = -1$ pole —including a piece proportional to $\mathcal{J}_3(s)$ — reverse their sign with respect to their internal counterparts and

$$\mathrm{Res}\,[\zeta_{\mathcal{M}\,\mathrm{ext}}(s;\alpha); s=-1] = -\mathrm{Res}\,[\zeta_{\mathcal{M}}(s;\alpha); s=-1], \tag{4.19}$$

as a result of which the net zeta function $\zeta_{\mathcal{M}}(s;\alpha) + \zeta_{\mathcal{M}\,\mathrm{ext}}(s;\alpha)$ is finite at $s = -1$ regardless of the α value. The same cancellation applies for the resudues at $s = 1$ and $s = 3$. Such a cancellation is typical of odd D's, and does not happen in $D = 2$ because the residue receives then a contribution from $\mathcal{J}_2(s)$, which maintains its sign. To be brief, we only have to worry about the residue at $s = 2$. It is quite immediate that it is α independent. The conclusion is that for a $3-D$ Klein-Gordon field defined in both the exterior and the interior region, the whole dependence of the vacuum energy on the α parameter is contained in the finite part of $\zeta_{\mathcal{M}}(s;\alpha) + \zeta_{\mathcal{M}\,\mathrm{ext}}(s;\alpha)$.

The α-dependences of the residue r_α and of the finite part p_α of $\zeta_{\mathcal{M}}(s;\alpha)$ at $s = -1$ are depicted in Figs. 2a and 2b for the internal modes only. The residue is simply formula (4.16), while the finite parts have been obtained through numerical evaluation of (4.13) by the methods described in refs. [29, 30]. Fig. 2c shows the inclusion of the external modes, and the net dependence on α of the vacuum energy. Though we do not pay too much attention to the absolute figures, but only to the dependence on α (in other words, to the derivative of the finite part with respect to α), it is worthwhile noting that the value at $\alpha = 0$ furnishes us with an opportunity to verify our results. We have obtained that the value of the graph at $\alpha = 0$ is $\frac{1}{a}0.005634... = 2 \cdot \frac{1}{a}0.002817...$ i.e. twice the figure found in [23] for an ordinary free field, as had to be expected.

5 $D = 2$ fermions

For $D = 2$ massless Dirac particles under the influence of the same magnetic field as in sect. 1 and 3, the Dirac equation reads

$$(i \not{\partial} + e \not{A})\Psi = 0, \tag{5.1}$$

with $\not{v} \equiv \gamma^\mu v_\mu$ ($\gamma^0 = \sigma^3$, $\gamma^1 = i\sigma^2$, $\gamma^2 = -i\sigma^1$), $A^0(\vec{r}) = 0$ and $\vec{A}(\vec{r})$ as in (1.2), (for previous works where ζ-function techniques are studied in fermionic systems see [35, 36, 37]).

The boundary conditions that we choose on the bag, given by the circle $r = a$, are those of the M.I.T. bag model

$$-i \not{n}\Psi = \Psi$$

where n stands for the normal vector.

It has been remarked in refs.[3, 2] that in this problem it would be too restrictive to impose regularity at the origin for the modes. If one imposes regularity the result is that the domain of the operator is not dense and, consequently, one loses self-adjointness. Making $\Psi(\vec{x}, t) = \psi(\vec{x})\, e^{-iEt}$, let us note the space-dependent part of a particular mode by

$$\psi(\vec{x}(r, \varphi)) = \begin{pmatrix} \chi^1(r) \\ \chi^2(r)\, e^{i\varphi} \end{pmatrix} e^{im\varphi}.$$

It is easily seen that if one demands regularity for the modes characterized by $m = -[\alpha] - 1$, then one is left only with the trivial solution $\Psi = 0$. As we advanced in the introduction, we have followed the analysis which was set forth in [10]; the outcome is that for the particular value of $m = -[\alpha] - 1$, one should choose the solution with regular χ^1 for positive α, and the one with regular χ^2 when α is negative. In any case this amounts to picking an element among a family of possible self-adjoint extensions for the Hamiltonian under the b.c. in question. Then, the whole set of Hamiltonian eigenfrequencies consists of:

1. the k's satisfying $f_\nu(ka) \equiv J^2_\nu(ka) - J^2_{\nu+1}(ka) = 0$ with $\nu = \{\alpha\} - 1$ if $\alpha > 0$, and with $\nu = -\{\alpha\}$ otherwise.
2. (a) the k's satisfying $f_{l+\{\alpha\}}(ka) = 0$, for $l = 0, 1, 2, \ldots$

   (b) the k's satisfying $f_{l+1-\{\alpha\}}(ka) = 0$, for $l = 0, 1, 2, \ldots$

where $\{\alpha\}$ denotes the fractional part of α.

It is then adequate to define

$$\zeta^{\rm f}_\nu(s) = \sum_{n=1}^{\infty} \lambda^{-s}_{\nu,n}\,, \text{ for Re } s > 1, \tag{5.2}$$

where $\lambda_{\nu,n}$ means the nth nonvanishing zero of $f_\nu(\lambda)$. Now we may write down the fermionic zeta function as

$$\zeta^{\rm f}_{\mathcal{M}}(s) \equiv \zeta^{\rm f}_{1\,\mathcal{M}}(s) + \zeta^{\rm f}_{2\,\mathcal{M}}(s), \tag{5.3}$$

where

$$\begin{aligned} \zeta^{\rm f}_{1\,\mathcal{M}}(s) &\equiv \theta(\alpha)\zeta^{\rm f}_{\{\alpha\}-1}(s) + \theta(-\alpha)\zeta^{\rm f}_{-\{\alpha\}}(s), \\ \zeta^{\rm f}_{2\,\mathcal{M}}(s) &\equiv \sum_{n=0}^{\infty} \zeta^{\rm f}_{\{\alpha\}+n}(s) + \sum_{n=0}^{\infty} \zeta^{\rm f}_{1-\{\alpha\}+n}(s). \end{aligned} \tag{5.4}$$

It follows that the Casimir energy will have to fulfil the equalities

$$\begin{aligned} E_C(-\alpha) &= E_C(\alpha), \\ E_C(\alpha+k) &= E_C(\alpha), \quad \text{for } \alpha>0, k=1,2,3,\dots \\ E_C(\alpha-k) &= E_C(\alpha), \quad \text{for } \alpha<0, k=1,2,3,\dots \end{aligned} \tag{5.5}$$

(compare with relations (3.2)). Since we have now this sort of periodicity when shifting α by integer values, it suffices for our study to take $0 \le \alpha < 1$.

With the help of the auxiliary object

$$\overline{\zeta^{\rm f}_{\mathcal{M}}}(s;\beta) \equiv a^s \sum_{l=0}^{\infty} \zeta^{\rm f}_{l+\beta}(s), \tag{5.6}$$

(analogous to (3.3) for bosons) we are able to express the complete zeta function as

$$\begin{aligned} \zeta^{\rm f}_{\mathcal{M}}(s;\alpha) &= a^s \zeta^{\rm f}_{\alpha-1}(s) + \overline{\zeta^{\rm f}_{\mathcal{M}}}(s;\alpha) + \overline{\zeta^{\rm f}_{\mathcal{M}}}(s;1-\alpha) \\ &= a^s[\zeta^{\rm f}_{\alpha-1}(s) + \zeta^{\rm f}_{\alpha}(s)] + \overline{\zeta^{\rm f}_{\mathcal{M}}}(s;1+\alpha) + \overline{\zeta^{\rm f}_{\mathcal{M}}}(s;1-\alpha) \end{aligned} \tag{5.7}$$

(For the numerical methods to be applied below, the second form proves to be more suitable around $\alpha = 0$).

Starting from the partial wave zeta function (5.2), we make use of the technique described in [26, 27] and find an analytic continuation to the domain $-1 < \text{Re } s < 0$ given by the integral representation

$$\zeta^{\rm f}_{\nu}(s) = \frac{s}{\pi} \sin\frac{\pi s}{2} \int_0^{\infty} dx\, x^{-s-1} \ln\left\{\pi x\, e^{-2x}[I^2_{\nu}(x) + I^2_{\nu+1}(x)]\right\}, \quad \text{for } -1 < \text{Re } s < 0. \tag{5.8}$$

For $\nu \neq 0$, and by a subtraction method similar to the one applied in the bosonic case, we obtain the more convenient form

$$\begin{aligned} \zeta^{\rm f}_{\nu}(s) = \nu^{-s}\frac{s}{\pi}\sin\frac{\pi s}{2} \Bigg[& \left\{ \frac{1}{s} B\left(\frac{s+1}{2}, -\frac{s}{2}\right) + 2^s B\left(\frac{s+1}{2}, -s\right) + 2^s B\left(\frac{s+3}{2}, -s\right)\right\}\nu \\ & + \mathcal{S}^{\rm f}_N(s,\nu) \\ & + \frac{1}{2s} B\left(\frac{s+1}{2}, -\frac{s}{2}\right) - \frac{1}{8} B\left(\frac{s+1}{2}, -\frac{s}{2}\right)\frac{1}{\nu} + \overline{\mathcal{J}}^{\rm f}_1(s)\frac{1}{\nu} + \sum_{n=2}^{N} \mathcal{J}^{\rm f}_n(s)\frac{1}{\nu^n}\Bigg], \end{aligned} \tag{5.9}$$

with

$$\begin{aligned} \mathcal{S}^{\rm f}_N(s,\nu) &\equiv \int_0^{\infty} dx\, x^{-s-1} \left\{ \ln\left[\mathcal{L}^{\rm f}(\nu,x)\right] - \sum_{n=0}^{N} \frac{\mathcal{U}^{\rm f}_n(t(x))}{\nu^n} \right\}, \\ \mathcal{L}^{\rm f}(\nu,x) &= [\sqrt{2\pi\nu}(1+x^2)^{1/4} e^{-\nu\eta(x)}]^2 \frac{1}{2}[I^2_{\nu}(\nu x) + I^2_{\nu+1}(\nu x)], \end{aligned} \tag{5.10}$$

where

$$\begin{aligned} \mathcal{U}^{\rm f}_0(t) &= \ln(1-t), \\ \mathcal{U}^{\rm f}_1(t) &= -\frac{t}{4} + \frac{t^3}{12}, \\ \mathcal{U}^{\rm f}_2(t) &= \frac{t^8}{8} + \frac{t^4}{8} - \frac{t^5}{8} - \frac{t^6}{8}, \\ \mathcal{U}^{\rm f}_3(t) &= \frac{5t^3}{192} + \frac{t^4}{8} + \frac{9t^5}{320} - \frac{t^6}{2} - \frac{23t^7}{64} + \frac{3t^8}{8} - \frac{179t^9}{576}, \\ \mathcal{U}^{\rm f}_4(t) &= \frac{t^4}{32} + \frac{17t^5}{128} - \frac{t^6}{8} - \frac{165\,t^7}{128} - \frac{37\,t^8}{64} + \frac{327\,t^9}{128} + \frac{57\,t^{10}}{32} - \frac{179\,t^{11}}{128} - \frac{71\,t^{12}}{64}, \\ &\vdots \end{aligned} \tag{5.11}$$

and

$$\begin{aligned}\overline{\mathcal{J}}^{\mathrm{f}}_1(s) &= \frac{1}{24}B\left(\frac{s+3}{2},-\frac{s}{2}\right),\\ \mathcal{J}^{\mathrm{f}}_n(s) &= \int_0^\infty dx\, x^{-s-1}\,\mathcal{U}^{\mathrm{f}}_n(t(x)).\end{aligned} \tag{5.12}$$

In order to include $\nu=0$ or close values, we also find the alternative representation

$$\begin{aligned}\zeta^{\mathrm{f}}_\nu(s) &= -\frac{1}{\pi}\left(\nu+\frac{1}{2}\right)^2\frac{1}{s+1}-\frac{1}{\pi}\left(\nu+\frac{1}{2}\right)^2(-1+\ln 2)+\frac{1}{\pi}\mathcal{R}^{\mathrm{f}}_\nu(-1)+\mathcal{O}(s+1),\\ \mathcal{R}^{\mathrm{f}}_\nu(s) &= \int_0^\infty dx\, x^{-s-1}\left\{\ln\left[\pi x e^{-2x}\left(I^2_\nu(x)+I^2_{\nu+1}(x)\right)\right]+\left(\nu+\frac{1}{2}\right)^2\frac{1}{\sqrt{x^2+1}}\right\},\end{aligned} \tag{5.13}$$

which is like (2.11), for bosons. Note the vanishing of the $s=-1$ pole for $\nu=-1/2$, which happens because $\zeta^{\mathrm{f}}_{\nu=-1/2}(s)=(\pi/2)^{-s}(2^s-1)\zeta_R(s)$ is finite at $s=-1$.

The analogue of expression (3.6) for the fermionic case is

$$\begin{aligned}\overline{\zeta^{\mathrm{f}}_{\mathcal{M}}}(s;\beta)=\frac{1}{\pi a}\quad&\left[\frac{1}{2}\zeta_H(-2,\beta)+\frac{1}{12}\zeta_H(0,\beta)-\left(\frac{97}{20160}+\frac{\pi}{256}\right)\zeta_H(2,\beta)+\left(\frac{13}{20160}+\frac{35\pi}{32768}\right)\zeta_H(3,\beta)\right.\\ &+\sum_{l=0}^\infty S^{\mathrm{f}}_4(-1,l+\beta)(l+\beta)\\ &+\left(-\frac{1}{12}+\frac{\beta}{4}+\frac{\pi}{128}-\zeta_H(-2,\beta)-\zeta_H(-1,\beta)\right)\left(\frac{1}{s+1}+\ln a-1\right)\\ &-\frac{1}{24}-\frac{\ln 2}{12}+\beta\frac{\ln 2}{4}-\frac{\psi(\beta)}{24}-\frac{\pi\psi(\beta)}{128}-(2+\ln 2)\,\zeta_H(-2,\beta)-(1+\ln 2)\,\zeta_H(-1,\beta)\\ &\left.-\zeta'_H(-2,\beta)-\zeta'_H(-1,\beta)-\frac{1}{4}\zeta'_H(0,\beta)+\mathcal{O}(s+1)\right].\end{aligned} \tag{5.14}$$

Actually, with this plus (5.7) and (5.13) we realize that

$$\zeta^{\mathrm{f}}_{\mathcal{M}}(s;\alpha)=\frac{1}{a}\left[\frac{1}{64}\left(\frac{1}{s+1}+\ln a\right)+q^{\mathrm{f}}_\alpha+\mathcal{O}(s+1)\right], \tag{5.15}$$

i.e. after adding up all the contributions, the residue of the resulting pole at $s=-1$ is independent of α, like the residues at $s=1$ and $s=2$, and, in consequence, the dependence of $E_{vac}(\Lambda)$ on α is given by $-\frac{1}{2a}q^{\mathrm{f}}_\alpha$ (remember the minus sign associated to Dirac particles, see for instance ref.[19]), where q^{f}_α is the remaining finite part of the ζ-function once the pole has been removed and does —predictably— depend on the magnetic field. Numerical values are given in table 2. The fact that $q^{\mathrm{f}}_1=q^{\mathrm{f}}_0$, comes from the equality $\zeta^{\mathrm{f}}_{\mathcal{M}}(s;0)=\zeta^{\mathrm{f}}_{\mathcal{M}}(s;1)$, which is in the end a consequence of the modified Bessel function identity $I_{-n}=I_n$, $n=1,2,3,\ldots$. The values of q^{f}_α are shown in Fig. 3a. For the sake of clarity we also give a plot for an extended domain of α which illustrates the particular periodicities of the fermionic case (5.5), see Fig. 3b.

6 Ending comments

A scalar Klein-Gordon field subject to Dirichlet boundary conditions and under the influence of an external magnetic field producing a single flux line has been studied in two- and three-dimensional spaces. We have obtained a nontrivial effect in the dependence of the vacuum energy on the flux which would be invisible (within the order of our approximation) without the presence of a finite-sized bag. Considering only the internal field modes in the $D=2$ case, we arrive at the conclusion that the vacuum energy undergoes a finite variation when

α	q^{f}_{α}
0	-0.00583
0.1	$+0.05603$
0.2	$+0.07735$
0.3	$+0.07753$
0.4	$+0.06656$
0.5	$+0.05045$
0.6	$+0.03314$
0.7	$+0.01733$
0.8	$+0.00484$
0.9	-0.00312
1	-0.00583

Table 2: Finite part q^{f}_{α} of the fermionic complete zeta function for varying reduced flux α.

the magnetic flux is changed. It is interesting to note that around $\alpha = 0$, the vacuum energy decreases when the flux grows. In this sense, the system would seem to energetically favour the presence of such fluxons.

In $D = 3$ the flux line is diametrically threading a sphere and we have quite a different resulting picture. Taking just the internal modes, we see that it is not true that the divergent pieces are independent of the magnetic flux. In fact we have seen that the coefficient giving the logarithmic divergence in Λ (or if you prefer, the residue of the ζ-function at $z = -1$) depends non-analytically on α. This is also the case for the coefficient giving a Λ^2 divergence, associated to the residue at $s = 1$. The inclusion of external modes dramatically modifies the situation. Their associated divergences exactly cancel those from the internal part, except the one arising from the pole at $s = 2$, but this piece does not contain any dependence on α. At $\alpha = 0$, our result agrees with the one found in ref. [23]. Around this point the system seems to oppose to the growth of the kind of fluxons we have pictured, in the sense that some amount of energy must be provided.

Fermions in $D = 2$ have also been considered. While the bosonic energy was periodic in α with period$= 1$, the fermionic one is —not too surprisingly— periodic with the same period only on each real semiaxis separately as shown in Fig. 3b, where q^{f}_{α} is represented. The divergences of $E_{reg}(\Lambda)$ are independent of α, with the same transparency in the physical interpretation of the result as in the $D = 2$ bosonic case. For values in a neighbourhood of $\alpha = 0$, the energy is seen to decrease as the absolute value of the flux grows. So we have that the fermionic case in $D = 2$ shares with the scalar one this property.

Refs. [25] include a study of the three-dimensional bag involving gauge (bosonic) and fermionic massless fields without external flux. By way of rough comparison with some of the figures obtained in these works we may evaluate the ratio between the maximum variation of the vacuum energy for Dirac field and the same quantity for a complex Klein-Gordon field. Taking into account that this variation is given by $\frac{0.0397}{a}$ for the fermionic case, and $\frac{0.0975}{a}$ for the scalar one, we have that the ratio is 0.41. In other words, the energy of a Klein-Gordon field is in this sense more sensitive to flux changes.

To finish this work we shall briefly comment how the analysis that we have performed in this article with models without a mass term carries over to cases where this term is present. Of course, had we incorporated a mass term, the result for the finite contributions to the zeta functions would be different and would call for some extra, though feasible, effort (see [11]). The divergent pieces would also change, but in a way which is quite trivial. In general, the residue of a pole at a point s, would be transformed into itself plus a linear combination of the residues of the poles at $s + 2k$ for positive integer k's, with coefficients given by even powers of the mass. In other words, if one finds that in the massless case divergences are α independent, this property carries over to the massive case.

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Figure Captions

Fig. 1. Bosonic zero-point energy E_C in $D = 2$ for $a = 1$ (then $E_C = q_\alpha$) as a function of the reduced flux α.
Fig. 2. Description of $\zeta_{\mathcal{M}}(s; \alpha)$ —for the $D = 3$ internal modes only— at $s = -1$, as a function of the reduced flux α: **a)** residue r_α, **b)** finite part p_α, **c)** comparison of p_α for internal and external modes, together with the Casimir energy E_C ($aE_C = p_{\alpha\,\mathrm{int}} + p_{\alpha\,\mathrm{ext}}$), in $D = 3$, where $p_{\alpha\,\mathrm{int}}$ is the same as in **b)**.
Fig. 3. **a)** Finite part of the fermionic zero-point energy $E_C = -\frac{1}{2a} q_\alpha^f$ in $D = 2$ for $a = 1$ as a function of the reduced flux α. **b)** Same as in **a**; we have simply enlarged the domain of α considered.

Fig. 1

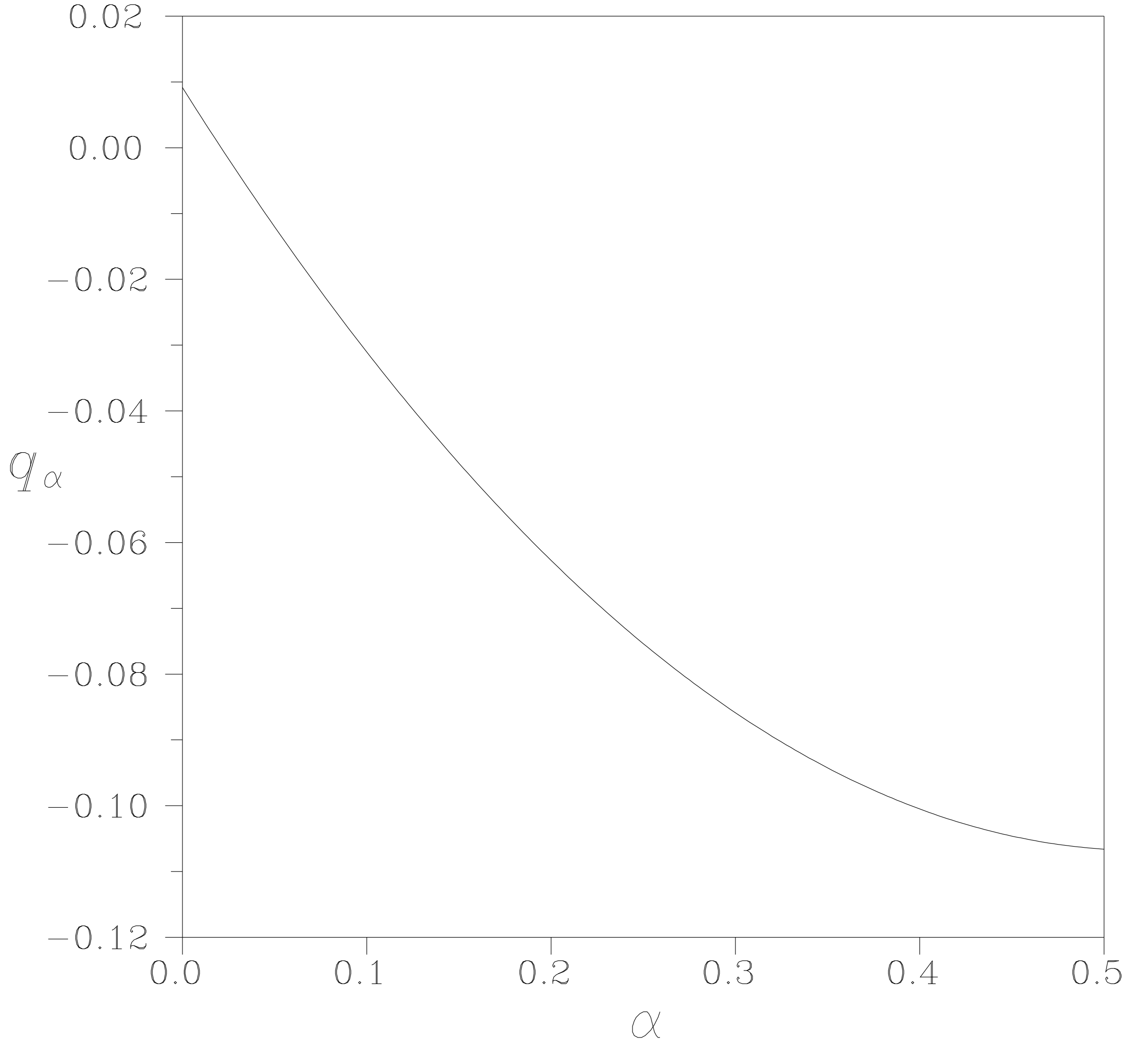


Fig. 2a

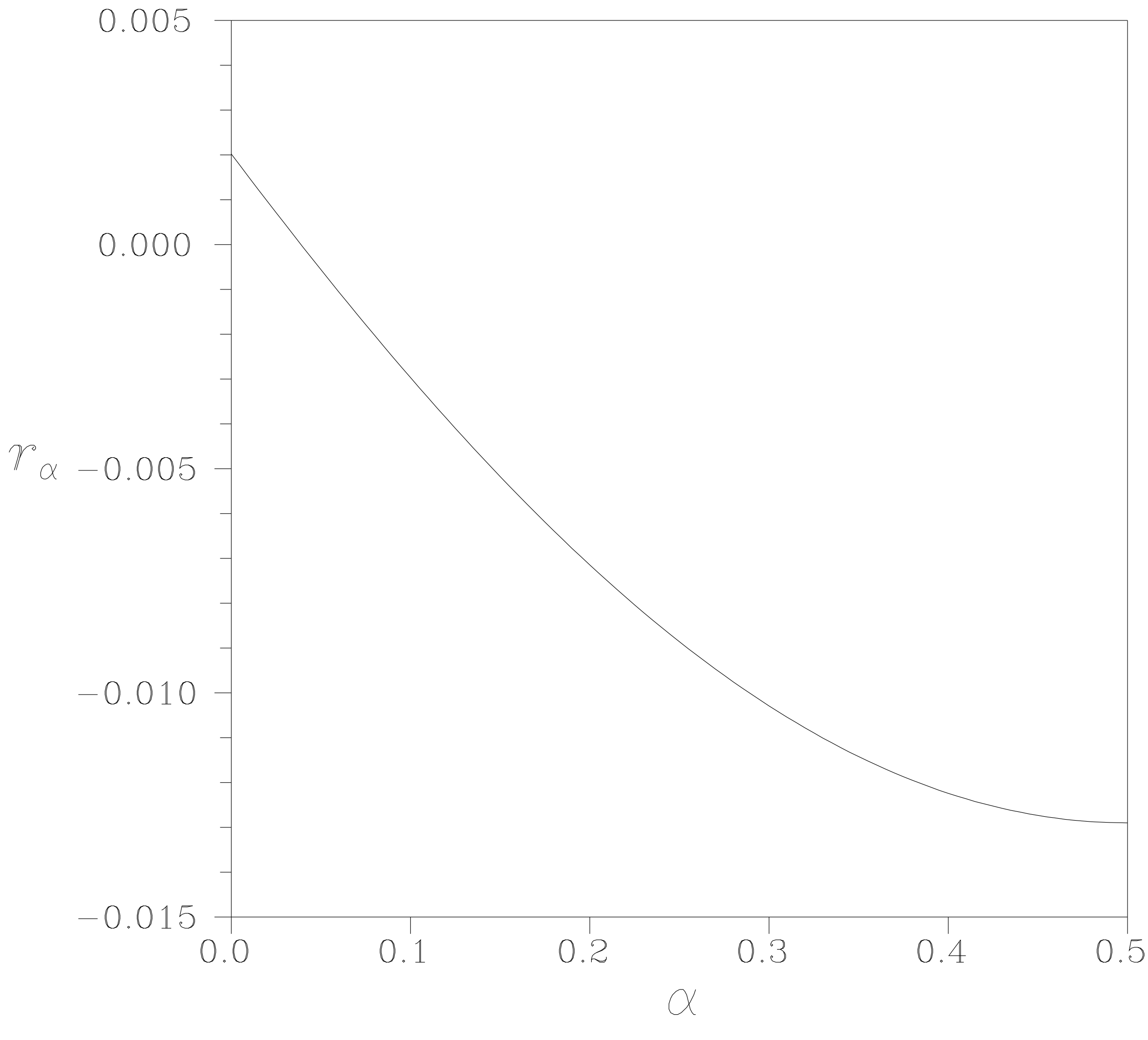
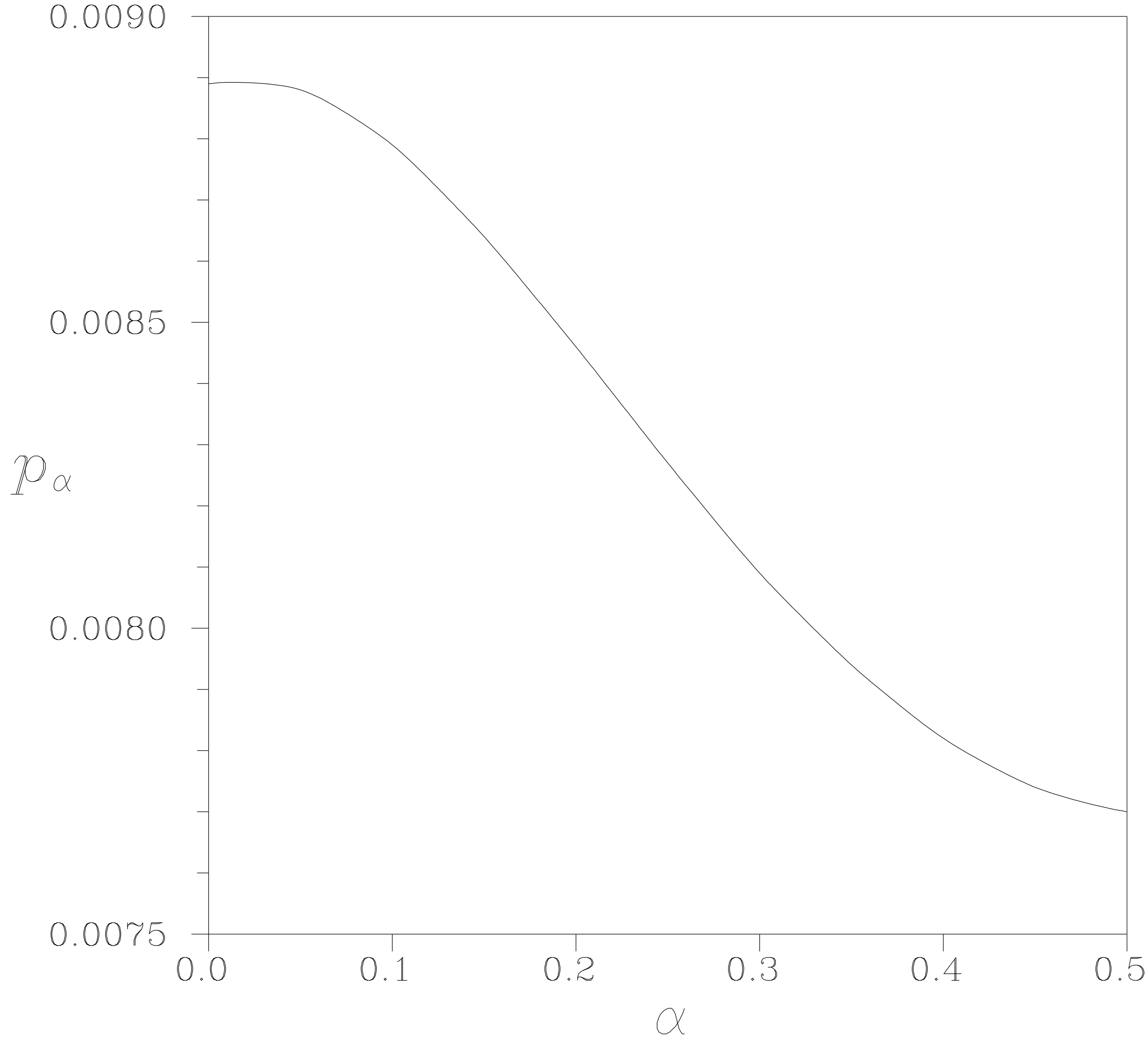


Fig. 2b



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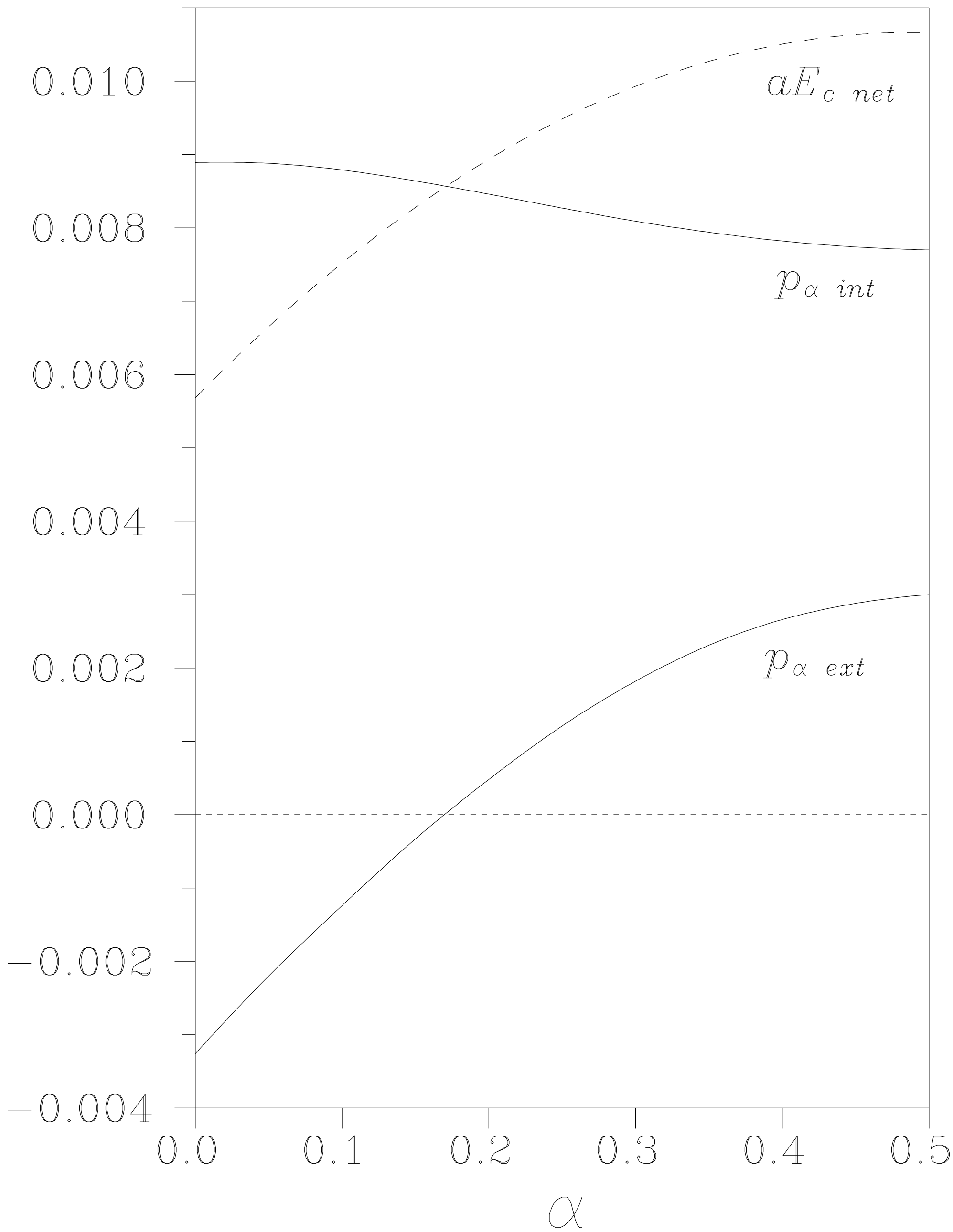

0.010
0.008
0.006
0.004
0.002
0.000
−0.002
−0.004
0.0
0.1
0.2
0.3
0.4
0.5
α
$\alpha E_{c\ net}$
$p_{\alpha\ int}$
$p_{\alpha\ ext}$

Fig. 3a

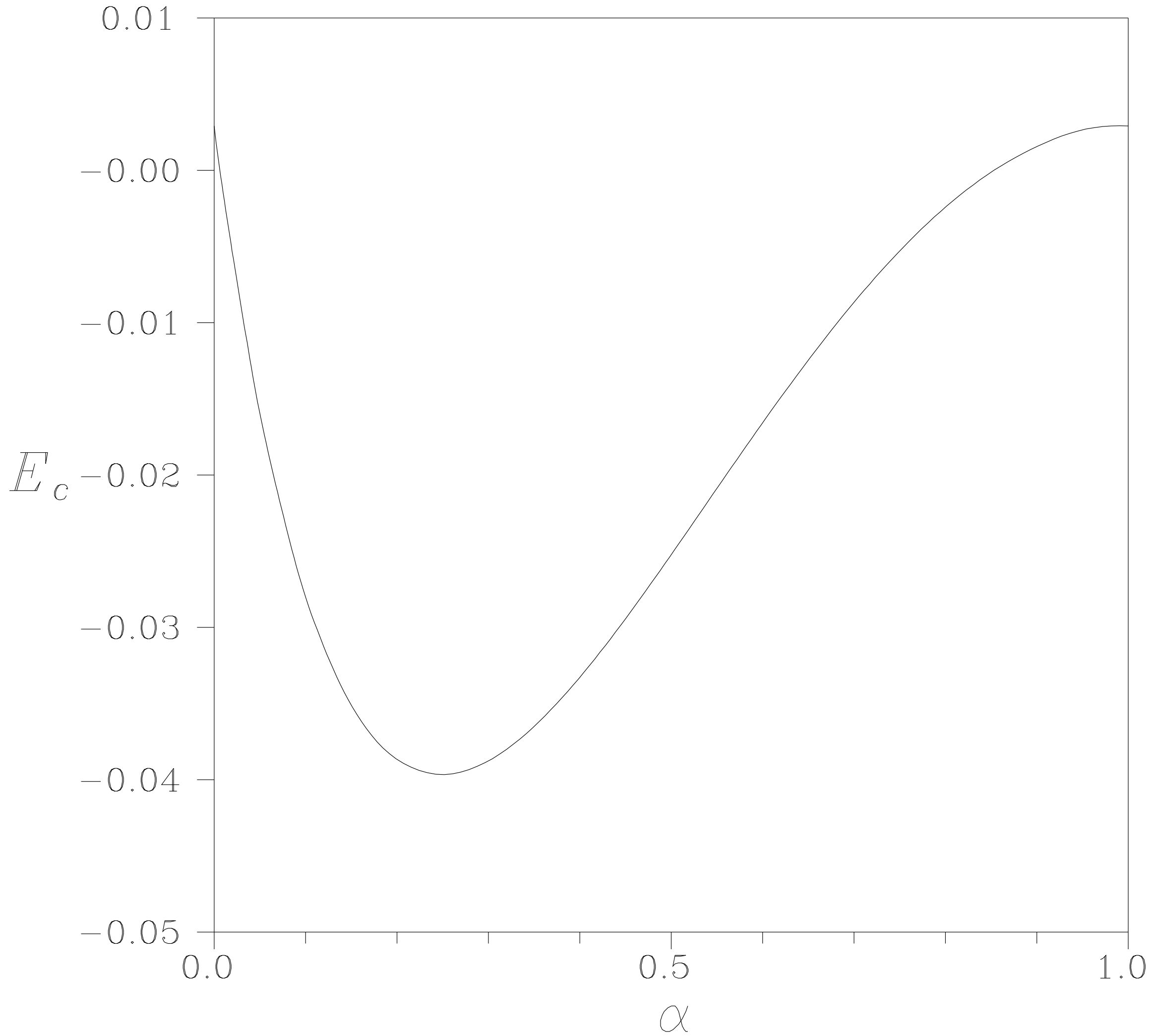

Fig. 3b